

### SOLUTION EXERCISE SHEET 3

**Exercise 1.** This is a consequence of the definition along with the Heine-Borel Theorem, which allow us to pick finitely many open sets to cover  $K$ .

Indeed, by normal convergence we have that, for each  $z \in K$ , there exists a  $\varepsilon_z > 0$  with  $\overline{D(z, \varepsilon_z)} \subset U$  such that

$$\sum_{k=0}^{\infty} \|f_k\|_{L^\infty(\overline{D(z, \varepsilon_z)})} < \infty.$$

On the other hand, we trivially have an open covering of  $K$  by simply considering the family of open discs  $\{D(z, \varepsilon_z)\}_{z \in K}$ . By compactness, we can choose a finite subcover  $K \subset \cup_{j=1}^m D(z_j, \varepsilon_j)$ . Therefore,

$$\sum_{k=0}^{\infty} \|f_k\|_{L^\infty(K)} \leq \sum_{k=0}^{\infty} \sum_{j=1}^m \|f_k\|_{L^\infty(\overline{D(z_j, \varepsilon_j)})} = \sum_{j=1}^m \sum_{k=0}^{\infty} \|f_k\|_{L^\infty(\overline{D(z_j, \varepsilon_j)})} < \infty.$$

You can justify switching the order of summation either by using the fact that each infinite sum is finite or the fact that each summand is non-negative. The proof is complete.

**Exercise 2.** By hypothesis, we have that, for any  $r \in (0, \varepsilon)$ ,

$$\sum_{k=0}^{\infty} |a_k| r^k < \infty, \quad \text{and} \quad \sum_{k=0}^{\infty} |b_k| r^k < \infty.$$

We must show that

$$\sum_{k=0}^{\infty} c_k (z - z_*)^k$$

converges, where

$$c_k = \sum_{\ell=0}^k a_\ell b_{k-\ell}.$$

Indeed, it is enough to see that, for any  $|z - z_*| \leq r$ , we have that

$$\begin{aligned} \sum_{k=0}^{\infty} |c_k| |z - z_*|^k &\leq \sum_{k=0}^{\infty} |c_k| r^k = \sum_{k=0}^{\infty} \left| \sum_{\ell=0}^k a_\ell b_{k-\ell} \right| r^k \\ &\leq \sum_{k=0}^{\infty} \sum_{\ell=0}^k |a_\ell| r^\ell |b_{k-\ell}| r^{k-\ell} \\ &\leq \left( \sum_{\ell=0}^{\infty} |a_\ell| r^\ell \right) \left( \sum_{k=0}^{\infty} |b_k| r^k \right) < \infty, \end{aligned}$$

having interchanged the order of summation and re-indexed  $k$  in the last line. This is possible, again, due to the non-negativity of the summands. The proof is complete.

**Exercise 3.** Let  $z_0 \in D(z_*, \rho)$  arbitrary. Define

$$\varepsilon := \frac{1}{2}(\rho - |z_0 - z_*|) \quad \text{and} \quad r := |z_0 - z_*| + \varepsilon < \rho.$$

Then, for any  $z \in \overline{D(z_0, \varepsilon)}$ ,

$$|z - z_*| \leq |z - z_0| + |z_0 - z_*| \leq \varepsilon + |z_0 - z_*| = r < \rho.$$

Therefore,

$$\sum_{k=0}^{\infty} |a_k| \|z - z_0\|_{L^\infty(\overline{D(z_0, \varepsilon)})}^k \leq \sum_{k=0}^{\infty} |a_k| r^k < \infty,$$

having used that  $r < \rho$ . Since  $z_0$  was arbitrary, we conclude the desired property.